$x = 6$, $y = 5$. The quadruple $(a, b, x, y) = (4, 1, 6, 5)$ satisfies the original system (1).

If, however, $a = 0$, the two-sided inequality (3) is fulfilled by two even integers $x = 2$ and $x = 4$. Plugging these into (1), the first of these two values yields contradiction, while the second one results in $(a, b, x, y) = (0, 0, 4, 4)$, which is a solution.

Taking symmetry into account (hence dismissing $x \geqslant y$), we obtain the following pairs (x, y): $(4, 4)$, $(6, 5)$ and $(5, 6)$.

858. This matrix can reveal the combinatorial nature of the problem:

$$\begin{bmatrix} i(A,D) & i(B,D) & i(C,D) \\ i(A,E) & i(B,E) & i(C,E) \end{bmatrix}$$

where

$$i(X,Y) = \begin{cases} 0 & \text{when } XY \leqslant 1 \\ 1 & \text{when } XY > 1 \end{cases}$$

(XY is the length of the segment with endpoints X, Y).

Since $AB = AC = BC = DE = 1$, the problem comes down to showing that at least one row or one column of this matrix has only zero entries. Suppose this is not the case. Then the following pattern appears in the matrix (up to a possible permutation of the symbols A, B, C and/or D, E, which does not influence the conditions of the problem): $\left[\begin{smallmatrix}1 & 1 & \\ & & 1\end{smallmatrix}\right]$; in terms of geometry (with notation as above):

$$AD > 1, \quad BD > 1, \quad CE > 1;$$

i.e.

$$AD > AC, \quad BD > BC, \quad CE > DE.$$

Let π be the perpendicular bisector plane of the segment CD. These inequalities imply that the points A and B lie on the same side of π as C, while E lies on the same side of π as D does (and no one of those points lies on π). Therefore π separates the triangle ABC from the segment DE, contradicting the condition that they meet. The result follows.

Are these distributed uniformly? ... that is on the V/V_{max} method

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In the XXI century, astronomers are obtaining *huge* amounts of observations with various level of specificity. Extracting knowledge from this *cosmos* of data obviously requires statistical analysis. Such analysis can be of different kinds: from extracting brightness and positions of objects from an image, through joint analysis of multiple observations in order to find a period of some phenomenon (e.g., eclipses in a stellar binary system), to analysis of data from various sources in order to determine parameters of a specific object (e.g., what is the distance to the Galactic center or what is the fraction of mass in the Universe that consists of barions).

Astronomy is significantly different from physics with respect to how the data to be analyzed are obtained: astronomy is based on observations of phenomena that we have no control of, whereas physics is mostly based on experiments performed (and hence controlled) by the scientist. This nature of astronomical observations poses a severe difficulty – sometimes increasing the sample of objects under study is extremely expensive and may require time that is longer than the expected lifetime of the researcher. Hence, astronomers often face incomplete samples of objects, even though they very much would like it to be otherwise. There are other obstacles, e.g., information about objects studied may come from observations taken under different conditions, epochs, etc. These subtle differences have to be taken into account during the statistical analysis, which is not an easy task.

In this article, I would like to present a statistical method that is typical for astronomy and bears an exotic name "V/V_{max}." It was designed in the late 60s in order to tackle the following issue: we have a catalog of quasars with known brightness and redshift and we would like to know if quasars are distributed uniformly in space.

Quasar (from "quasi-stellar object") is a type of active galaxy that emits extremely bright radiation.

Note that for the most luminous quasars, the sample can be considered complete for very large distances, whereas for the less luminous quasars the sample is complete for smaller distances. At the first "glance" the quasar space density may seem to be getting lower with increasing distance from Earth due to different luminosities. However, this may be an artifact of higher overall completeness for smaller distances (compare Fig. 1 and Fig. 2).

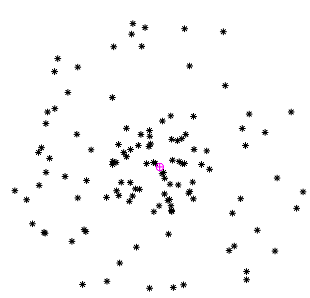

Fig. 1. Illustration of 150 quasars observed by a 2D astronomer in a 2D Universe. These quasars seem to show concentration around Earth (marked by the ⊕ symbol).

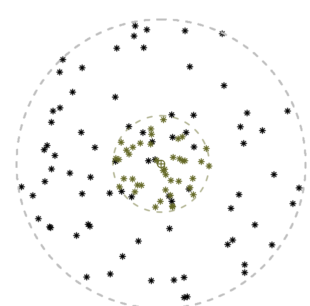

Fig. 2. Illustration of the same quasars as in Fig. 1 divided into two groups with different absolute magnitudes. Quasars from each group are uniformly placed in the area in which they can be seen.

Let us start with basics and assume that all quasars have the same absolute brightness (i.e., they are equally bright when observed at a distance of 10 parsecs). If all quasars are uniformly distributed in space, then the *observed* quasars are uniformly distributed in the part of space, a ball, in which we can see them (see black points in Fig. 2). The volume of this ball is labeled $V_{\max}$. For each quasar K we determine the distance and calculate the volume of the ball V_K with radius equal to that distance. For quasars uniformly distributed in space, the distribution of $V/V_{\max}$ values should be uniform in $[0, 1]$ interval (Fig. 3), hence, the mean of $V/V_{\max}$ should be approximately equal to 1/2. If the distribution of $V/V_{\max}$ does not satisfy these conditions, then we can reject the hypothesis of uniform distribution of quasars in space.

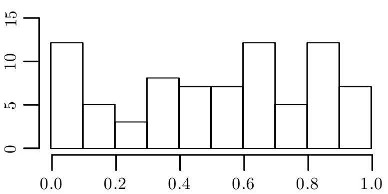

Fig. 3. Histogram of $V/V_{\max}$ for "black quasars" from Fig. 2 assuming fixed value of $V/V_{\max}$. The height of each bar corresponds to the number of quasars with $V/V_{\max}$ value in a range defined by the base of the bar.

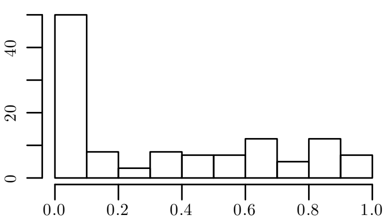

Fig. 4. Histogram of $V/V_{\max}$ for all quasars from Fig. 2 assuming fixed value of $V_{\max}$.

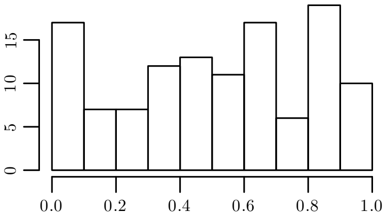

Fig. 5. Histogram of $V/V_{\max}$ for all quasars from Fig. 2 with $V_{\max}$ estimated for each quasar separately.

What about quasars with different absolute brightness (i.e. as with real quasars)? First, let us assume (for simplicity) that for each quasar, the absolute brightness has one of the two values (like in Fig. 1 and 2). As already noted, it will affect the sample of observed quasars because the more luminous quasars will be seen from larger distances than the less luminous ones. If we use the same $V_{\max}$ value for each quasar, then we will get a histogram that is similar to Fig. 4, which is significantly different from the one in Fig. 3. We have to modify our calculations and determine the $V_{\max}$ value separately for each quasar, based on the absolute brightness (labeled $V_{\max,K}$ for quasar K). Nonetheless, if all quasars are distributed uniformly in space, then for each quasar the value of $V_K/V_{\max,K}$ can be treated as randomly drawn from $[0, 1]$ interval, independently from other quasars. In other words, the distribution of $V_K/V_{\max,K}$ should still be uniform in $[0, 1]$ interval (see Fig. 4 and 5) and its mean should be close to 1/2. These conditions are fulfilled both by all quasars in Fig. 2, as well as only the black ones and only the colored ones.

Now we are ready for a generalization – each quasar has different absolute brightness. For each quasar we can calculate corresponding V_K and $V_{\max,K}$. Once more we expect that for a uniform quasar distribution the $V/V_{\max}$ distribution will be uniform in $[0, 1]$ range and with mean close to 1/2. If we have observational data, then it is enough to calculate $V/V_{\max}$, check if the distribution is consistent with expectations and we know whether or not quasars are uniformly distributed!

The $V/V_{\max}$ method has been further developed in various directions. For Curious Readers (that already have some familiarity with these matters) I would like to briefly present an extension that is most important in my opinion. It is possible to analyze how parameters of populations change with time. Let us imagine the Universe at the age corresponding to a redshift z. Let $\rho(z)$ be the ratio of volume density of studied objects then and now (i.e., $z = 0$). Instead of considering volume V we consider generalized volume:

$$V'(z) = \int_0^z \rho(z')dV(z'),$$

where $V(z)$ is volume in co-moving coordinates and is calculated based on the assumed cosmological model. For each quasar we know its redshift z_K, hence, we can calculate the corresponding $V'_K(z_K)$ and $V'_{\max,K}(z_K)$. If we assume that quasars are distributed uniformly and we assume $\rho(z)$ function that is close to the true one, then the distribution of $V'/V'_{\max}$ should be... (I do not think that I need to repeat myself). How we can know what $\rho(z)$ we should assume? The easiest approach is to check many different possibilities, for each of them calculate $V'/V'_{\max}$, and at the end select those that give the results expected.

It is worth noting that the $V/V_{\max}$ method and its variants are still used in scientific papers. For example, a few years ago this method was used for a completely different problem: studying exoplanet frequency $f(q)$ as a function of star-planet mass ratio q for very small values of q, i.e., for cases for which our knowledge is rather poor ([1], [2]). The problem was how to include planets found in data collected in heterogeneous way. Instead of $\rho(z)$, the searched function was $f(q)$ and instead of $V(z)$ researchers used probability of finding in given system a planet with a different mass ratio. Detailed description of these studies and their results is a topic for a separate article...

[1] Udalski et al. *OGLE-2017-BLG-1434Lb: Eighth $q < 1 \times 10^{-4}$ Mass-Ratio Microlens Planet Confirms Turnover in Planet Mass-Ratio Function.* Acta Astronomica 68.1 (2018): 1-42.

[2] Jung et al. *KMT-2017-BLG-0165Lb: A Super-Neptune-mass Planet Orbiting a Sun-like Host Star.* The Astronomical Journal 157.2 (2019): 72.